Continuous Data – Master Equations

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Revision 01

DESIGN VERIFICATION

Normal Distribution: Probability Density Function (PDF)

$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{(x-u)}{\sigma}\right)}$$

Introduction to Statistical Quality Control Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. (1996). ISBN 0-471-30353-4. Page 57.

Normal Distribution: Cumulative Probability Density Function (CDF) – (Area under the Curve)

$$\Phi(a) = P(x \le a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Introduction to Statistical Quality Control Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. (1996). ISBN 0-471-30353-4. Page 58.

Fraction Defective

<u>One-Sided, Lower Design Requirement (LDR)</u>

$$\Phi(LDR) = P(x \le LDR) = \int_{-\infty}^{LDR} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{LDR-\bar{x}}{\sigma}\right)^2} dx$$

% Defective =Φ(LDR) *100

PPM Defective $_{\text{Below LDR}}$ = Φ(LDR) *1,000,000

<u>One-Sided, Upper Design Requirement (UDR)</u>

$$1-\Phi(UDR) = 1 - P(x \le UDR) = \int_{UDR}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{UDR-\bar{x}}{\sigma}\right)^2} dx$$

% Defective = (1 - Φ(UDR)) *100

PPM Defective $_{\text{Above UDR}}$ = (1-Φ(UDR)) *1,000,000

% Defective $_{\text{Below LDR}}$ + % Defective $_{\text{Above UDR}}$ = Total Estimated % Defective

STATISTICAL CONFIDENCE INTERVAL OF THE MEAN

One-Sided Confidence Interval of the mean, Lower Design Requirement (LDR) or Upper Design Requirement (UDR)

$$\mu \geq \bar{x} - (Z_C)\left(\frac{s}{\sqrt{n}}\right)$$

Use this to determine the lower bound of the confidence bound – important when comparing the sample mean to a LDR

$$\mu \leq \bar{x} + (Z_C)\left(\frac{s}{\sqrt{n}}\right)$$

Use this to determine the upper bound of the confidence interval – important when comparing the sample mean to a UDR

Two-Sided Confidence Interval of the Mean, Lower & Upper Design Requirements (LDR & UDR)

$$\bar{x} - (Z_{C/2})\left(\frac{s}{\sqrt{n}}\right) \leq \mu \leq \bar{x} + (Z_{C/2})\left(\frac{s}{\sqrt{n}}\right)$$

Use this to determine the lower and upper bounds of the confidence interval simultaneously – important when comparing the sample mean to both a LDR and a UDR

Applied Statistics and Probability for Engineers. Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. (2003). ISBN 0-471-20454-4. Page 250.

DESIGN VERIFICATION

STATISTICAL TOLERANCE INTERVALS FOR THE NORMAL DISTRIBUTION

One-Sided Tolerance Interval

$$\mathrm{P}(R_L < \Phi(LTL \le x \le \infty) < 1) = 1 - \alpha \quad \text{OR} \quad \mathrm{P}(R_L < \Phi(\infty \le x \le UTL) < 1) = 1 - \alpha$$

$$LTL = \bar{x} - k_1(s)$$
$$UTL = \bar{x} + k_1(s)$$

$$K_1 = \frac{\mathrm{Z_p} + \sqrt{\mathrm{Z_p^2} - \left(1 - \frac{\mathrm{Z_C^2}}{2(\mathrm{n}-1)}\right)\left(\mathrm{Z_p^2} - \frac{\mathrm{Z_C^2}}{\mathrm{n}}\right)}}{1 - \frac{\mathrm{Z_C^2}}{2(\mathrm{n}-1)}}$$

Natrella 1963, *Experimental Statistics*, NBS Handbook 91, US Department of Commerce.

n =	sample size
p =	minimum proportion covered by tolerance interval
α =	probability of coverage
Z_p =	z-value for minimum covered proportion of distribution
Z_α =	z-value for alpha

Two-Sided Tolerance Interval

$$\mathrm{P}(R_L < \Phi(LTL \le x \le UTL) < 1) = 1 - \alpha$$

$$LTL = \bar{x} - k_2(s)$$
$$UTL = \bar{x} + k_2(s)$$

$$K_2 = Z_{(1+p)/2}\sqrt{\frac{\nu\left(1 + \frac{1}{n}\right)}{\chi^2_{\alpha,\nu}}}$$

Howe 1969, *"Two-sided Tolerance Limits for Normal Populations - Some Improvements"*, Journal of the American Statistical Association, 64 , pages 610-620. https://www.itl.nist.gov/div898/handbook/prc/section2/prc263.htm

n =	sample size
v =	n-1 (aka degrees of freedom)
p =	minimum proportion covered by tolerance interval
α =	1-α = probability of coverage
X^2 =	chi-square value for α and v degrees of freedom
$Z_{(1+p)/2}$ =	z-value for cumulative probability (1+p)/2

IDCF for χ^2 Distribution: $\mathrm{P} = F_{(x,\nu)} = \int_0^x \frac{t^{(\nu-2)/2}e^{-t/2}}{2^{\nu c/2}\,\Gamma(\nu/2)}$

Algebraic Approximation of χ^2: $\chi^2 = \left[\left(\left(Z_\alpha\sqrt{\frac{2}{9\upsilon}}\right) + \left(1 - \left(\frac{2}{9\upsilon}\right)\right)\right)^3\right] * (\upsilon)$

Wilson, E. B., & Hilferty, M. M. (1931). The distribution of chi-square. Proceedings of the National Academy of Sciences of the United States of America, 17, 684-688.

PROCESS VALIDATION

Process Capability Indices for Process Validation

Two-Sided Process Specification

$$C_p = \frac{USL - LSL}{6\hat{\sigma}}$$

Cp is a rate of process capability. Here the estimated standard deviation comes from the data measured from one sample (sample group / lot / batch/ etc.). This metric does not consider if the sample distribution is centered between the LSL and USL or not.

Introduction to Statistical Quality Control Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. ©1996. ISBN 0-471-30353-4. Page 438.

$$C_pk = min\left(\frac{USL - \hat{\mu}}{3\hat{\sigma}}, \frac{\hat{\mu} - LSL}{3\hat{\sigma}}\right)$$

Cpk is a rate of process capability. Here the estimated standard deviation comes from the data measured from one sample (sample group / lot / batch/ etc.). This metric does consider if the distribution of the sample taken is centered between the LSL and USL or not.

Introduction to Statistical Quality Control Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. ©1996. ISBN 0-471-30353-4. Page 443.

One-Sided Process Specification

$$C_{p-lower} = \frac{\mu - LSL}{3\hat{\sigma}}$$

$$C_{p-upper} = \frac{USL - \mu}{3\hat{\sigma}}$$

Introduction to Statistical Quality Control Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. ©1996. ISBN 0-471-30353-4. Page 440.

Confidence Intervals for Process Capability Indices

2 sided "1-α" Confidence Interval for Cp

$$C_P \le \widehat{C_p}\left(\sqrt{\frac{\chi^2_{(\alpha/2)(,n-1)}}{n-1}}\right)$$

Upper Bound of 2 sided "1-α" confidence interval for Cp

$$C_P \ge \widehat{C_p}\left(\sqrt{\frac{\chi^2_{(1-\alpha/2),(n-1)}}{n-1}}\right)$$

Lower Bound of 2 sided "1-α" confidence interval for Cp

Introduction to Statistical Quality Control Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. (1996). ISBN 0-471-30353-4. Page 447.

2 sided "1-α" Confidence Interval for CpK

$$C_{pk} \ge \hat{C}_{pk}\left[1 - z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}{}^2} + \left(\frac{1}{2(n-1)}\right)}\right]$$

Lower Bound of 2 sided "1-α" confidence interval for CpK

$$C_{pk} \le \hat{C}_{pk}\left[1 + z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}{}^2} + \left(\frac{1}{2(n-1)}\right)}\right]$$

Lower Bound of 2 sided "1-α" confidence interval for CpK

Introduction to Statistical Quality Control Third Edition. Douglas C. Montgomery. John Wiley & Sons, Inc. (1996). ISBN 0-471-30353-4. Page 448.

Probability of Acceptance for Lot Sampling – AQL an RQL

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(z)^2}{2}}$$

Standard Normal Distribution:
Probability Density Function (PDF)
($\mu=0$, $\sigma^2=1$) and $z=(x-\mu)/\sigma$

Statistics 5th Edition. William L. Hays. Wadsworth Group / Thompson Learning. ©1994. ISBN 0-03-074467-9. Page 238.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x)^2}{2}} dx$$

Standard Normal Distribution:
Cumulative Density Function (CDF)

https://www.itl.nist.gov/div898/handbook/eda/section3/eda3661.htm

$$\varphi_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

General Normal Distribution
Probability Density Function (PDF)

Statistics 5th Edition. William L. Hays. Wadsworth Group / Thompson Learning. ©1994. ISBN 0-03-074467-9. Page 238.

$$\Phi_{\mu,\sigma}(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

General Normal Distribution:
Cumulative Density Function (CDF)

CRC Standard Mathematical Tables 25th Edition. William H Beyer. CRC Press Inc. ©1979 ISBN 0-8493-0625-6 page 521

LOT ACCEPTANCE SAMPLING

Probability of Acceptance for Lot Sampling – AQL an RQL

$$P(x \leq L) = Pa = \Phi\left(\sqrt{n}\left(Z_{1-p} - k\right)\right) \quad \text{where,} \quad k = \frac{\left(Z_{1-\alpha}\right)\left(Z_{1-RQL}\right) + \left(Z_{1-\beta}\right)\left(Z_{1-AQL}\right)}{\left(Z_{1-\alpha}\right) + \left(Z_{1-\beta}\right)}$$

Duncan (1986). *Quality Control and Industrial Statistics, 5th edition*

$$\Phi\left(\sqrt{n}\left(Z_{1-p} - k\right)\right) = P\left(x \leq \left(\sqrt{n}\left(Z_{1-p} - k\right)\right)\right) = \int_{-\infty}^{\sqrt{n}\left(Z_{1-p}-k\right)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\left(\sqrt{n}\left(Z_{1-p}-k\right)\right)-\mu}{\sigma}\right)^2} d_{\sqrt{n}\left(Z_{1-p}-k\right)}$$

General Form Probability of Acceptance at a percent defective = p

$$\Phi\left(\sqrt{n}\left(Z_{1-AQL} - k\right)\right) = P\left(x \leq \left(\sqrt{n}\left(Z_{1-AQL} - k\right)\right)\right) = \int_{-\infty}^{\sqrt{n}\left(Z_{1-AQL}-k\right)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\left(\sqrt{n}\left(Z_{1-AQL}-k\right)\right)-\mu}{\sigma}\right)^2} d_{\sqrt{n}\left(Z_{1-p}-k\right)}$$

Probability of Acceptance at a percent defective = AQL

$$\Phi\left(\sqrt{n}\left(Z_{0.5} - k\right)\right) = P\left(x \leq \left(\sqrt{n}\left(Z_{0.5} - k\right)\right)\right) = \int_{-\infty}^{\sqrt{n}\left(Z_{0.5}-k\right)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\left(\sqrt{n}\left(Z_{0.5}-k\right)\right)-\mu}{\sigma}\right)^2} d_{\sqrt{n}\left(Z_{1-p}-k\right)}$$

Probability of Acceptance at a percent defective = IQL

$$\Phi\left(\sqrt{n}\left(Z_{1-RQL} - k\right)\right) = P\left(x \leq \left(\sqrt{n}\left(Z_{1-RQL} - k\right)\right)\right) = \int_{-\infty}^{\sqrt{n}\left(Z_{1-RQL}-k\right)} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\left(\sqrt{n}\left(Z_{1-RQL}-k\right)\right)-\mu}{\sigma}\right)^2} d_{\sqrt{n}\left(Z_{1-p}-k\right)}$$

Probability of Acceptance at a percent defective = RQL